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On the Swendsen–Wang Dynamics. II. Critical Droplets and Homogeneous Nucleation at Low Temperature for the Two-Dimensional Ising Model

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We consider the Swendsen–Wang dynamics for the two-dimensional Ising model at low temperature in the presence of a small negative magnetic field h and with plus boundary conditions at the boundary of an arbitrarily large square. We analyze in detail the tunneling from the metastable phase to the stable one. In particular, we obtain an upper bound on the tunneling time t_β by explicitly constructing paths in the space of spin configurations that drive the system from the metastable phase to the stable one. In our analysis the transition takes place through the formation of droplets of the right phase inside the wrong one with side greater than a certain critical value l_c. The values of the tunneling time and of l_c coincide with those found for a single-spin-flip dynamics in finite volume by Jordao-Neves and Schonmann.

KEY WORDS: Ising model; random cluster dynamics; stable and metastable phases; nucleation; critical droplets.

INTRODUCTION

In this paper we continue the systematic analysis of the Swendsen–Wang (SW)[1] dynamics started in ref. 2. SW is a random cluster dynamics reversible with respect to the Gibbs measure for the Ising model. A constructive definition of the algorithm is provided at the beginning of Section 1 and for a discussion of its main features the reader is referred to ref. 2.

We shall investigate the ferromagnetic two-dimensional Ising model when the inverse temperature β is very large and the external magnetic field

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h is arbitrarily small. In ref. 2 we were able to show, for the above situation, the exponential convergence to equilibrium in the particular case of boundary conditions parallel to the external magnetic field. For the case of boundary conditions opposite to h it turned out that a much more delicate analysis was needed involving a detailed study of the metastable behavior of the system. This analysis is the content of the present paper.

As is well known, the Ising model at low temperature and zero magnetic field exhibits a phenomenon of coexistence of two different phases with opposite nonzero magnetization. The configurations of the system have a kind of "symmetric double-well structure" even in the thermodynamic limit. The two wells are related to the typical configurations characterizing the two opposite phases. For small h this features is in some sense preserved even though now the well corresponding to the phase parallel to h becomes deeper. The phase opposite to the field becomes metastable and does not correspond anymore to a true equilibrium state of the system on a macroscopic scale. The magnetic field h decides the phase no matter how small it is, but its effects become relevant only on a suitable scale diverging when $h \to 0$, in the sense that only on large scales does the volume energy dominate the surface energy. From a dynamical point of view this means that if one starts with a typical configuration of the metastable phase (i.e., the majority of the spins are opposite to the field), then locally the system will undergo only "small fluctuations" around the false equilibrium for a certain amount of time (large if β/h is large) until it will "tunnel" to the true equilibrium. This "tunneling" is a local phenomenon consisting in the homogeneous formation, through the whole volume, of many droplets (nuclei) of the new stable phase inside the metastable phase (homogeneous nucleation). The main physical feature of this transition is the existence of a critical value $l_c(h)$ for the size of the droplets. Droplets whose side is smaller then $l_c(h)$ have the tendency to shrink, whereas the larger ones have the tendency to grow and there is an "activation energy" necessary to create them.

In a recent paper Jordao-Neves and Schonmann[3] studied a single-spin-flip Glauber dynamics for the two-dimensional Ising model in a finite box with small, say positive, magnetic field, showing the metastable behavior in the limit β tending to ∞. They were able to prove that the transition from the configuration with all spins opposite to the field to the configuration with all spins parallel to the field happens with a probability tending to 1 in the limit of large β via the formation of a critical square "droplet" full of plus spins. Moreover, they found values for the critical size as well as for the typical time of the transition which are in agreement with what can be found on heuristic grounds using energetic arguments.

In the present paper, in the case of SW dynamics, we describe the nucleation phenomenon in an arbitrarily large (infinite) volume at very low temperature with plus boundary conditions and negative magnetic field. We find that locally the system has a behavior very similar to the one detected by Jordao-Neves and Schonmann for the Glauber dynamics (of range one). As a consequence of this analysis, we show, in particular, a result on the absence of persistence of large sets of spins with magnetization oposite to the field. This is the crucial point which is needed to prove, even in this case, the exponential decay to the equilibrium (see Theorem 2 of ref. 2; see also ref. 4 for the case of a single-spin-flip dynamics). More specifically, we show that, due to the homogeneous nucleation, after a sufficiently large time the negatively magnetized stable phase is formed in the whole bulk, whereas the influence of the plus boundary conditions has an effect only at a short distance from the boundary. We stress that the appearance of the new phase is a consequence of the simultaneous formation of many large enough droplets of minus spins throughout the volume and that the tendency of these droplets to survive and to grow under the dynamics is due to an essentially local mechanism.

The main result of the present paper is summarized in the following theorem. Let Q_L be the square of edge L in $\mathbb{Z}^2$ centered at the origin. We denote by $\sigma \in \{-1, +1\}^{Q_L}$ a generic spin configuration in Q_L and by σ_t its evolution at time t according to the SW dynamics on Q_L with plus boundary conditions (see Section 1). Let, for any $A \subset Q_L$, $\mathscr{A}_t$ be the event

$$\mathscr{A}_t = \{\exists \sigma \in \{-1, +1\}^{Q_L} \text{ such that } \sigma_t(x) = +1 \, \forall x \in A\}$$

Theorem. There exists $c > 0$ such that, given $h < 0$ sufficiently small in absolute value, there exist constants $\beta_0(h)$, $L_0(h)$ such that if $\beta > \beta_0$, $L > L_0(h)$, $t \geqslant \exp[\beta(4/|h| + c)]$; then

$$P(\mathscr{A}_t) \leqslant \exp[-k(\beta)|A|]$$

provided $\text{dist}(A, \partial Q_L) > L_0(h)$, where $k(\beta) \to \infty$ as $\beta \to \infty$.

Clearly, using the uniformity in the initial configuration, we only need to prove the above bound for $t = t_\beta$ with

$$t_\beta = \exp\left[\beta\left(\frac{4}{|h|} + c\right)\right]$$

In the rest of the paper we set

$$\mathscr{A} = \mathscr{A}_{t_\beta}$$

1. DEFINITIONS AND SKETCH OF THE PROOF

Let us first give some notation and a constructive definition of the SW algorithm for the two-dimensional Ising model with external magnetic field h and + boundary conditions.

Let $\Lambda \subset \mathbb{Z}^2$ be a finite box; any subset $C \subset \Lambda$ such that any two points $x, y \in C$ can be connected by a nearest neighbor (n.n.) path of sites in C is a "cluster."

By $C_{\{x_1,...,x_n\}}$ we will denote the cluster consisting of exactly the sites $x_1,..., x_n$.

At each integer time t, to each bond b in Λ, $b=(x, y)$, $|x-y|=1$, in Λ we associate a bond variable $n_b(t)$ and a random number $v_b(t)$ uniformly distributed in $[0, 1]$; to each cluster $C \subset \Lambda$ we associate a random variable $\xi_C(t)$ uniformly distributed in $[0, 1]$.

Given a spin configuration at time t: $\{\sigma_t(x)\}_{x\in\Lambda} \in \{-1, +1\}^\Lambda$ we construct a configuration σ_{t+1} with the following rule:

1. Determination of the bond variables $b_{(x,y)}(t+1)$:

$$\begin{aligned}
&\text{if} \quad \sigma_t(x) \neq \sigma_t(y), \quad \text{then} \quad n_{(x,y)}(t+1) = -1 \\
&\text{if} \quad \sigma_t(x) = \sigma_t(y), \quad \text{then} \quad n_{(x,y)}(t+1) = -1 \qquad \text{if} \quad v_{(x,y)}(t) < e^{-\beta} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad n(x, y)(t+1) = 1 \qquad \text{if} \quad v_{(x,y)}(t) \geqslant e^{-\beta}
\end{aligned}$$

2. Determination of the new configuration: consider the clusters such that $x, y \in C$ if and only if there exists a connected path of bonds $b_1,..., b_n$ going from x to y with $n_{b_k}(t+1) = 1$ $\forall k = 1,..., n$. Then

$$\begin{aligned}
\sigma_{t+1}(x) = +1 \quad \forall x \in C \quad &\text{if} \quad C \cap \partial\Lambda \notin 0 \quad \text{and in this case we set } C = C_\infty \\
\text{or} \quad &\text{if} \quad \xi_C(t) \leqslant (1 + e^{-\beta h|C|})^{-1} \\
\sigma_{t+1}(x) = -1 \quad \forall x \in C \quad &\text{if} \quad \xi_C(t) > (1 + e^{-\beta h|C|})^{-1}
\end{aligned}$$

where $|C| \equiv \#\{x \in C\}$.

Let us now fix some more notation.

(a) Events in our basic probability space are denoted by capital script letters, e.g., $\mathscr{A}$, $\mathscr{B}$, $\mathscr{D}$,....

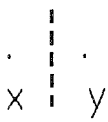
(b) For any $x \in \Lambda$, L even Q_x^L will denote the intersection between the square of side L, centered in x, and the box Λ.

(c) An event $\mathscr{A}$ will be called (x, L)-cylindrical if $\mathscr{A}$ depends only on the random variables $\{v_b(t), \xi_C(t)\}_{t\geqslant 0}$ for b and C inside Q_x^L.

Remark. If $\mathscr{A}_1$, $\mathscr{A}_2$ are (x_i, L_i)-cylindrical, $i = 1, 2$, and $\text{dist}(x_1, x_2) > (L_1 + L_2)/2$, then clearly

$$P(\mathscr{A}_1 \cap \mathscr{A}_2) = P(\mathscr{A}_1)\, P(\mathscr{A}_2)$$

(d) By "cut of the bond b" we denote the occurrence of the event $\{v_b(t) < e^{-\beta}\}$. The cut of the bond $b = (x, y)$ will be graphically represented by



(e) If $\omega = \{v_b(t),\ \xi_C(t)\}_{b,C,t}$ denotes a realization of the basic random process, we denote by $S_t\omega$ the shifted realization:

$$(S_t\omega) = \{v_b(t+s), \xi_C(t+s)\}_{b,C,s}$$

Similarly, if $\mathscr{A}$ denotes an event, then we set

$$S_t\mathscr{A} = \{\omega;\ S_t\omega \in \mathscr{A}\}$$

Let us now give a sketch of the strategy. The main idea behind the proof of the theorem is to show that:

1. Among the n points of the set A there are at least $\frac{1}{4}n$ sites $x_{i_1},..., x_{i_k}$, $k > n/4$, with $\text{dist}(x_{i_l}, x_{i_m}) > L$, $\forall l,\ m \in [1,..., k]$, such that within the time $t_\beta \approx \exp\{4\beta/|h|\}$ the dynamics, with large probability, has been able to grow a large droplet of minuses around x_{i_j}, $j = 1,..., k$, e.g., $Q^{2L_0}_{x_{i_j}}$, with $L \gg L_0 \gg l_c$, using only the variable $\{v_b(t), \xi_C(t)\}_t$ with b and C entirely inside $Q^{4L_0}_{x_{i_j}}$, $j = 1,..., k$.
2. Once the droplet $Q^{2L_0}_{x_{i_j}}$ of minuses is formed at time $\tau_{i_j} < t_\beta$, then it is "stable" under the dynamics for a time scale much larger than the critical time t_β.

In turn, the growth of the droplet $Q^{2L_0}_{x_{i_j}}$ will be decomposed into two parts corresponding to two different regimes:

(a) The *subcritical regime*, namely when the dynamics builts up a droplet of -1 of size $l_c(h) = [2/|h|] + 3$ around x_{i_j}. The time scale involved in this process is of order $\exp\{\beta(2l_c(h))\} \approx t_\beta$. For a precise definition of t_β see Section 2, (2.16).

(b) The *supercritical regime*, when, starting from the droplet of size $l_c(h)$ around x_{i_j} the dynamics enlarges it up to $Q^{2L_0}_{x_{i_j}}$. The time scale involved in this second process is quite independent of the exact value of $L_0 > l_c$, and it is of order $\exp(4\beta)$, provided L_0 is independent of β.

Let us now make the above ideas more precise. In Section 2.3 we will explicitly construct for any $x \in \Lambda$ with $\operatorname{dist}(x, \partial\Lambda) \geqslant 4L_0$ an event $\mathscr{E}_x$ such that:

(i) $\mathscr{E}_x$ is $(x, 4L_0)$-cylindrical (1.1)

(ii) $P(\mathscr{E}_x) \to 1$ as $\beta \to \infty$ (1.2)

(iii) The event

$$\mathscr{E}_x \cap \{\forall t < t_\beta, \nexists C; C \cap Q_x^{2L_0} \neq \emptyset, |C| > L_0; C = + \text{ at time } t\} \tag{1.3}$$

implies the event

$$\{\exists \tau_x < t_\beta; \sigma_{\tau_x}(y) = -1\ \forall y \in Q_x^{2L_0}\} \cap \{\sigma_{t_\beta}(x) = -1\}$$

for any initial configuration σ.

With the help of the event $\mathscr{E}_x$ we will now define the events $\mathscr{B}$ and $\mathscr{D}$ as follows:

$$\begin{aligned}\mathscr{B} \equiv \{&\exists \text{ at least } \tfrac{3}{4}n \text{ points } \{x_{i_k}\} \in \Lambda \\ &\text{such that } \forall k \text{ and } \forall t < t_\beta, \nexists C \text{ such that} \\ &C \cap Q_{x_{i_k}}^{2L_0} \neq \emptyset, |C| > L_0, C = + \text{ at time } t\}\end{aligned} \tag{1.4}$$

$$\begin{aligned}\mathscr{D} \equiv \{&\exists \text{ at least } \tfrac{3}{4}n \text{ points } \{x_{i_j}\} \in \Lambda \text{ such that } \mathscr{E}_{x_{i_j}} \\ &\text{occurred for any } j\}\end{aligned} \tag{1.5}$$

Then we write

$$P(\mathscr{A}) \leqslant P(\mathscr{A} \cap \mathscr{B} \cap \mathscr{D}) + P(\mathscr{B}^C) + P(\mathscr{D}^C) \tag{1.6}$$

where $\mathscr{A}$ is the event defined at the end of the introduction. Using (1.1)–(1.5), we get immediately that the first term in the rhs of (1.6) is zero. The second and third terms are estimated in the following two lemmas.

Lemma 1.1. For any β large enough,

$$P(\mathscr{B}^c) \leqslant 2^n \exp\left(-\beta \frac{|h|}{16} L_0 n + \frac{n}{2}\right)$$

Lemma 1.2. For any $L > L_0 > l_c(h)$ and β large enough,

$$P(\mathscr{D}^c) \leqslant e^{-k(\beta)n/4}$$

with

$$k(\beta) = \log\left[\frac{1}{2^4(1 - P(\mathscr{E}_x))}\right]$$

and $k(\beta) \to \infty$ as $\beta \to \infty$ because of the crucial property

$$P(\mathscr{E}_x) \to 1 \qquad \text{as} \quad \beta \to \infty$$

It is clear that (1.6) together with the lemmas prove the main theorem. The proof of the lemmas, based on rather simple expansions, similar to the low-temperature expansions of statistical mechanics, is given in Appendix A.

2. CONSTRUCTION OF THE EVENT $\mathscr{E}_x$: THE SUBCRITICAL REGIME

The section and the next one are devoted to the construction of the event $\mathscr{E}_x$ with the properties described in Section 1.

For simplicity, we take $x = 0$. Next we introduce two time scales:

$$\begin{aligned} T_2 &\equiv [\exp(2\beta + \delta\beta)], \qquad 0 < \delta \ll |h| \\ T_4 &\equiv [\exp(4\beta + \delta\beta)], \qquad 0 < \delta \ll |h| \end{aligned} \tag{2.1}$$

and a sequence of time intervals:

$$\begin{aligned} I_j^2 &\equiv [(4L_0)^2(j-1)T_2, (4L_0)^2 jT_2) \\ I_j^4 &\equiv [(4L_0)^2(j-1)T_4, (4L_0)^2 jT_4), \qquad j = 1, 2,... \end{aligned} \tag{2.2}$$

The introduction of these time scales is justified by the following heuristic considerations: if at time $t = 0$ we have a region of -1 like the one of Fig. 1, then the time necesary to observe a variation in the shape of

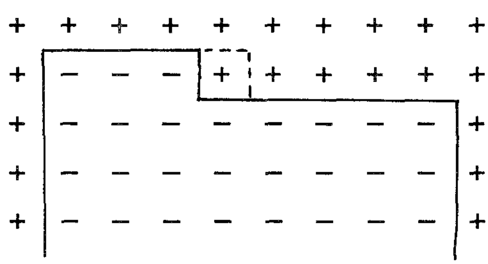


Fig. 1

our droplet is of order $\exp(2\beta)$, since we need to cut at least two bonds (the dashed ones in Fig. 1, for example), provided that the whole droplet does not flip to $+1$ before. This last hypothesis is satisfied with large probability if $|h|$ times the area of the droplet is >2.

However, the time necessary to enlarge, say, a square l by l of -1 spins to, e.g., a rectangle l by $(l+1)$ is of order $\exp(4\beta)$. This is so because one has first to create a protuberance like the one involed in Fig. 2, and this requires a cutting of four bonds (the dashed ones) and subsequently, with successive cutting of only two bonds, the protuberance expands to a full new line. This second part of the process requires only a time of order $\exp(2\beta)$ and thus the dominant time scale is $\exp(4\beta)$.

Given the above time scales, we will construct the set $\mathscr{E}_0$ as

$$\mathscr{E}_0 = \{\exists j, 0 \leqslant j < t_\beta \exp(-2\beta - 2\delta\beta) \text{ such that } \mathscr{E}^{\text{sub}}(j) \cap \mathscr{E}^{\text{super}}(j) \text{ holds}\}$$

where $\mathscr{E}^{\text{sub}}(j)$ and $\mathscr{E}^{\text{super}}(j)$ are two events such that for any j:

(a) $\mathscr{E}^{\text{sub}}(j)$, $\mathscr{E}^{\text{super}}(j)$ are $(0, 4L_0)$-cylindrical.

(b) $\mathscr{E}^{\text{sub}}(j)$ depends only on the process

$$\{v_b(t), \xi_C(t)\} \qquad \text{for} \quad t < j \cdot T_2$$

(c) $\mathscr{E}^{\text{super}}(j)$ depends only on the process

$$\{v_b(t), \xi_C(t)\} \qquad \text{for} \quad t \in (jT_2, t_\beta)$$

(d) If $\{v_b(t), \xi_C(t)\} \in \mathscr{E}^{\text{sub}}(j) \cap \mathscr{B}_0$, where $\mathscr{B}_0$ is given by

$$\mathscr{B}_0 \equiv \{\exists t \leqslant t_\beta; \text{at time } t \exists C = + \text{with } |C| \geqslant L_0 \text{ and } C \cap Q_0^{2L_0} \neq \varnothing\}$$

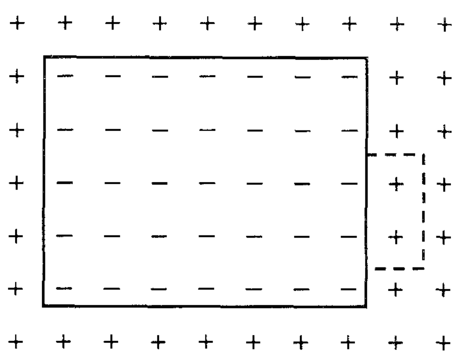


Fig. 2

then for any $\sigma \in \{-1, 1\}^{\Lambda}$

$$\sigma_{j \cdot T_2}(x) = -1 \qquad \forall x \in Q_0^{l_c}$$

(e) If $\{v_b(t), \xi_C(t)\}_{t \in [0, t_\beta]} \in \mathscr{E}^{\text{super}}(0) \cap \mathscr{B}_0$, then for any $\sigma \in \{-1, 1\}^{\Lambda}$ such that $\sigma(x) = -1 \ \forall x \in Q_0^{l_c}$ we have

$$\sigma_{t_\beta}(x) = -1 \qquad \forall x \in Q_0^{2L_0}$$

(f) We have

$$P(\mathscr{E}_0) \to 1 \qquad \text{as} \quad \beta \to \infty$$

In this section we will construct only $\mathscr{E}^{\text{sub}}(j)$; the construction of $\mathscr{E}^{\text{super}}(j)$ is postponed to Section 3.

Definition. The interval I_1^2 is good if:

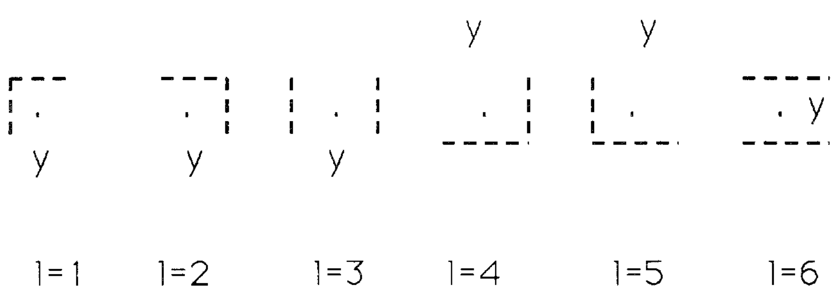
(a) $\nexists s \in I_1^2$ such that there are three or more cuts at time s in $Q_0^{4L_0}$.

(b) $\nexists s \in I_1^2$ such that there are two cuts at time s and $\exists C \subset Q_0^{4L_0}$ with $C = +$ at s.

(c) $\nexists s \in I_1^2$ such that there are two cuts at time s and $\exists C \subset Q_0^{4L_0}$ with $C = +$ at s.

(c) $\nexists s \in I_1^2$ such that there exists a $C \subset Q_0^{4L_0}$, $C = +$ at time s and $|h|\ |C| \geqslant 3$.

(d) $\forall i \leqslant (4L_0)^2\ \forall y \in Q_0^{4L_0}$, $\exists s_l(i, y) \in [(i-1)T_2, iT_2]$, $l = 1,..., 6$, such that at time s_l we have the cuts:



and $C_{\{y\}} = -$.

(e) At time $(4L_0)^2 T_2$ there are no clusters $C \subset Q_0^{4L_0}$, $C = +$.

An analogous definition holds for I_j^2, $j > 1$. The main points behind the above definition are that: (1) if the interval I_1^2 is good, then it has a tendency to enlarge and regularize the shape of the droplet of minuses in

$Q_0^{4L_0}$ of the initial configuration σ, and (2) most of the intervals I_j^2 are good in the sense that

$$P(I_j^2 \text{ is good}) \to 1 \qquad \text{as} \quad \beta \to 0$$

More precisely:

Lemma 2.1. Let R_{l_1,l_2} be a rectangle with sides l_1, l_2 such that $R_{l_1,l_2} \subset Q_0^{2L_0}$ and let $A \subset R_{l_1,l_2}$ be a cluster enjoing the property $\mathscr{P}$, where $\mathscr{P}$: (a) the smallest rectangle containing A is R_{l_1,l_2}, (b) it is not possible to disconnect A into parts with only one cut.

Then if $|A|\,|h| \geqslant 3$, if I_1^2 is good, if $\mathscr{B}_0$ holds, and if $\sigma(x) = -1\ \forall x \in A$, we have

$$\sigma_{(4L_0)^2 T_2}(x) = -1 \qquad \forall x \in R_{l_1,l_2}$$

Lemma 2.2. We have

$$P(I_1^2 \text{ is not good}) \leqslant e^{-\beta k}$$

with $k = |h| - 2\delta$.

Proof of Lemma 2.1. Let us first prove that $\sigma_t(x) = -1\ \forall x \in A$ $\forall t \leqslant (4L_0)^2 T_2$. Assume that there exists an $x_0 \in A$ and $s \in I_1^2$ such that $\sigma_{s-1}(x) = -1\ \forall x \in A$ and $\sigma_s(x_0) = +1$. Then either $\sigma_s(x) = +1\ \forall x \in A$ or at time s there were at least two cuts in $Q_0^{4L_0}$ and a cluster $C = +$ with $C \subset Q_0^{4L_0}$. Both cases are impossible using (b) and (c) of the definition of I_1^2 good, respectively, since, by assumption, $|h|\,|A| \geqslant 3$.

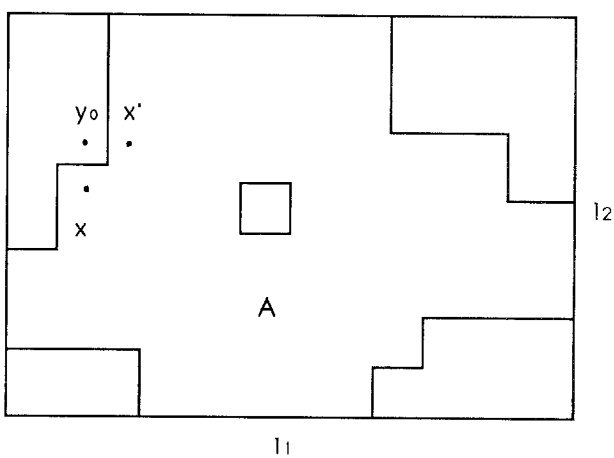


Fig. 3

Let us now prove that during a good interval I^2, the cluster of minuses in R_{l_1,l_2} grows under the dynamics, up to the complete covering of R_{l_1,l_2}.

Let $y_0 \in R_{l_1,l_2} \backslash A$ be such that $\exists x$, $x' \in A$, with $|x - y_0| = |x' - y_0| = 1$.

If A is different from R_{l_1,l_2}, such a point must exist by hypothesis.

The situation could be the one depicted in Fig. 3. We know that there exists a time $s \in [0, T_2]$ such that we see the cuts of Fig. 4 with $C_{\{y_0\}} = -1$. If at time $s-1$, $\sigma_{s-1}(y_0) = +1$, then clearly at time s, $\sigma_s(y_0) = -1$. If already $\sigma_{s-1}(y_0) = -1$, then the set $A_1 = A \cup \{y_0\}$ at time $s-1$ enjoys the same property as A and so, by previous arguments, it cannot decrease at time s. In both cases $A_1 = A \cup \{y_0\}$ enjoys the same properties as A and it is stable.

We now pick a new point $y_1 \in R_{l_1,l_2} \backslash A_1$ with the same property of y_0 and repeat the argument. By iterating this argument $l_1 l_2 - |A|$ times, we fill up the whole R_{l_1,l_2} with minus spins and the lemma is proved.

Proof of Lemma 2.2. It is easily seen that

$$\begin{aligned} P(I^2, \text{is not good}) &\leqslant (4L_0)^4 T_2 e^{-3\beta} + (4L_0)^2 T_2 k(L_0) e^{-2\beta - \beta|h|} \\ &\quad + (4L_0)^2 T_2 k(L_0) e^{-3\beta} + (4L_0)^4 6[1 - e^{-2\beta}(1 - e^{-\beta|h|})]^{T_2} \\ &\quad + k(L_0) e^{-\beta|h|} \leqslant e^{-(|h| - 2\delta)\beta} \end{aligned}$$

if $0 < \delta < |h|/3$ and β is sufficiently large, where $k(L_0) = \#\{$connected clusters in $Q_0^{4L_0}\}$.

The next step requires us to understand how much a sequence of "bad" (not good) intervals I_j^2 may affect a rectangular droplet of minuses.

Unfortunately, in order to solve the above problem, one has to distinguish among various situations in order to discard the most unlikely ones.

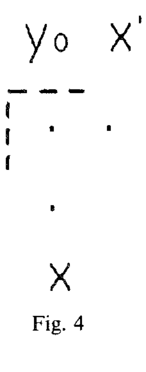


Fig. 4

Definitions

$\mathscr{F}_n^{[0,N]} \equiv \{\exists j_0, 0 \leqslant j_0 \leqslant N-n; I_j^2 \text{ is "bad" } \forall j, j_0 \leqslant j \leqslant j_0 + n\}$

$\mathscr{G}_n^{[0,N]} \equiv \{\exists j_0, 0 \leqslant j_0 \leqslant N-n;$ in the time interval $S_n = \bigcup_{j \geqslant j_0}^{j_0+n} I_j^2$ at least one of the following events occurs: (a) $\exists s_1 \in S_n$; at time s_1 we have more than 3 cuts in $Q_0^{4L_0}$ and a cluster $C \subset Q_0^{4L_0}$, $C = +$; (b) $\exists s_1, s_2 \in S_n$, $s_1 \leqslant s_2$; at time s_i, $i = 1, 2$, there are at least 3 cuts and a cluster $C \subset Q_0^{4L_0}$, $C = +\}$

$\mathscr{H}_{n,\gamma}^{[0,N]} \equiv \{\exists j, 0 \leqslant j \leqslant N-n; \#\{s \in I_j^2 \cup \cdots \cup I_{j+n}^2$ in which there are $\geqslant 2$ cuts and at least a cluster $C \subset Q_0^{4L_0}$, $C = +$; or $\geqslant 3$ cuts$\} > \gamma\}$

$\mathscr{M}_r^{[0,N]} \equiv \{\exists s \in I_1^2 \cup \cdots \cup I_N^2$; at time s there exist $C_1,..., C_k \subset Q_0^{4L_0}$, $C_i = +$, $\forall i = 1 \cdots k, \sum_{i=1}^k |C_i| \geqslant r\}$

It is quite clear that the complements of the events $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$, and $\mathscr{M}$ give some control on the loss of minus spins during the time interval $[0, N(4L_0)^2 T_2]$. This justifies the introduction of the following event:

$$\mathscr{R}_{n,\gamma,r}^{[0,N]} \equiv (\mathscr{F}_n^{[0,N]})^c \cap (\mathscr{G}_n^{[0,N]})^c \cap (\mathscr{H}_{n,\gamma}^{[0,N]})^c \cap (\mathscr{M}_r^{[0,N]})^c \qquad (2.3)$$

Proposition 2.1. Let $\sigma \in \{-1, 1\}^\Lambda$ be such that $\sigma(x) = -1$ $\forall x \in R_{l_1,l_2} \subset Q_0^{2L_0}$ with $(l_1 - 1)(l_2 - 1) \geqslant r_0$.

Suppose that $\mathscr{B}_0 \cap \mathscr{R}_{n,\gamma,r}^{[0,N]}$ holds with $r = r_0$. $\gamma = (l_1 - 2) \wedge (l_2 - 2)$, n, N arbitrary; then if I_j^2 is "good," for some $j \leqslant N$ we have

$$\sigma_{j(4L_0)^2 T_2}(x) = -1 \qquad \forall x \in R_{l_1,l_2}$$

Remark. In other words, if $\mathscr{B}_0 \cap \mathscr{R}_{n,\gamma,r}^{[0,N]}$ occurs with n, γ, r, N as in the proposition, then the dynamics not only cannot destroy the cluster of minuses R_{l_1,l_2}, but it is even able to reconstruct it.

Notice that this result is stronger than Lemma 2.1, since we allow in the sequence $I_1^2 \cup \cdots \cup I_N^2$ some "bad" intervals I_i^2.

Proof of Proposition 2.1. The proposition clearly follows from Lemma 2.2 if we can prove that at the end of an arbitrary sequence $S_{n'} = I_{j_0}^2 \cup \cdots \cup I_{j_0+n'}^2$ in $\bigcup_{j=1}^N I_j^2$ with $n' < n$ such that I_j^2 is "bad" $\forall_j$, $j_0 \leqslant j \leqslant j_0 + n'$ and $I_{j_0-1}^2$, I_{j+n+1}^2 are "good," then the set

$$\tilde{A} \equiv \{x; \sigma_{(j_0+n)(4L_0)^2 T_2}(x) = -1\} \cap R_{l_1,l_2}$$

contains a cluster A with the property $\mathscr{P}$ of Lemma 2.2.

Suppose that at the beginning of the sequence the set $\tilde{A}$ coincides with R_{l_1,l_2}. Since $\mathscr{R}_{n,\gamma,r}^{[0,N]} \cap \mathscr{B}_0$ holds, we have that in the sequence S_n we have at most γ times $s_1,..., s_\gamma$ such that at s_i we have exactly two cuts and one cluster $C=+$, $C\subset Q_0^{4L_0}$, and at most one time s^1 with exactly three cuts and one $C=+$, $C\subset Q_0^{4L_0}$.

Thus, during the sequence S_n we can lose at most $\gamma+2$ minus spins in R_{l_1,l_2}. In this estimate we have taken into account the fact that with three cuts we can create a site x which can be disconnected from the rest of the droplet of minuses by only a single cut. This site is considered to be lost anyway (see Fig. 5).

From the above considerations it follows that the cluster of minuses inside $R_{l_1l_2}$ has an area greater than or equal to $l_1l_2-\gamma-2\geqslant(l_1-1)(l_2-1)$ and thus its spins can never flip to $+1$ all together because of our choice of r and the definition of $\mathscr{M}_r^{[0,N]}$. Since $\gamma=(l_1-2)\wedge(l_2-2)$, it also follows that the clusters of minuses enjoy property $\mathscr{P}$ of Lemma 2.2 and the proposition follows. ∎

The next step is obviously an estimate from below of the probability of the event $\mathscr{R}_{n,\gamma,r}^{[0N]}$ for a given choice of the parameters N, n, γ, r. This can be done with the help of the next lemma, whose elementary proof is postponed to Appendix B.

Lemma 2.3. Given n, γ, r, if β is large enough, we have, for any $N\geqslant n$:

(a) $P(\mathscr{F}_n^{[0N]})\leqslant Ne^{-k\beta n}$ with $k=|h|-2\delta$.

(b) $P(\mathscr{G}_n^{[0N]})\leqslant 2Nn^2e^{2\beta-\beta|h|}k(L_0)$, where $k(L_0)$ is as in the proof of Lemma 2.2.

(c) $P(\mathscr{H}_{n,\gamma}^{[0N]})\leqslant Ne^{-\beta(|h|-2\delta)\gamma}$ with $\delta<|h|/2$.

(d) $P(\mathscr{M}_r^{[0N]})\leqslant N(4L_0)^2\,T_2k(L_0)e^{-|h|r\beta}$.

It follows from the above estimates and from Proposition 2.1 that if we start at time $t=0$ with a rectangular region of minus spins $R_{l_1l_2}\subset Q_0^{2L_0}$ with

$$|h|\,(l_1-1)(l_2-1)>2+|h|\,(l_1\wedge l_2)$$

and $l_1\wedge l_2<l_c(h)$, see (2.7), this droplet will persist with only small fluctuations with large probability up to a time τ_{l_1,l_2} of the order of

$$\tau_{l_1l_2}\approx NT_2 4L_0^2P((\mathscr{R}_{n,\gamma,r}^{[0,N]})^c\cup\mathscr{B}_0^c)^{-1} \tag{2.4}$$

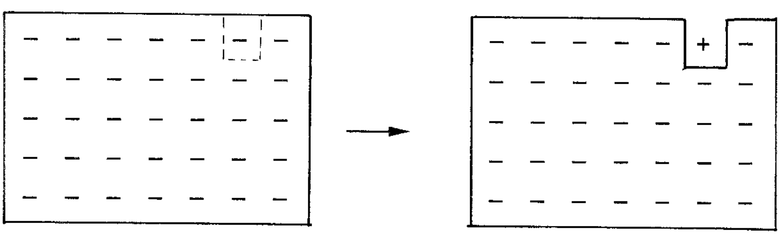


Fig. 5

with

$$n = 3(l_1 \wedge l_2)$$
$$\gamma = (l_1 - 2) \wedge (l_2 - 2)$$
$$r = (l_1 - 1)(l_2 - 1)$$
$$N = \exp\left(\beta\left\{2 \wedge \left[(|h| - 3\delta)(l_1 - 2) \wedge (l_2 - 2) + \frac{\delta}{2}\right]\right\}\right)$$

with these values of the parameters we get that, by Lemma 2.3,

$$P((\mathscr{R}_{n,\gamma,r}^{[0N]})^c) \leqslant e^{-\delta\beta} \to 0 \qquad \text{as} \quad \beta \to \infty$$

It follows from the proof of Lemma 1.1 in Appendix A that $P(\mathscr{B}_0^c)$ goes to zero as $\beta \to \infty$ much faster than $P(\mathscr{R}_{n,\gamma,r}^{[0N]})$ if L_0 is taken, e.g., equal to $L_0 = [1/|h|^3]$.

We thus get the important result that the *typical resistance time* (up to small fluctuations concentrated at the corners) of a droplet of minus spins $R_{l_1 l_2}$ with

$$|h|(l_1 - 1)(l_2 - 1) > 2 + |h|(l_1 \wedge l_2) \tag{2.5}$$

is of order

$$\tau_{l_1 l_2} \approx 4L_0^2 e^{2\beta} e^{\beta(2 \wedge \{(|h| - 3\delta)[(l_1 - 2) \wedge (l_2 - 2)] + \delta/2\})} \tag{2.6}$$

It remains to discuss the mechanism responsible for the growth of the cluster of minus spins.

Given the square Q_0^l, l even, we set $(x_i y_i)$, $i = 1, 2, 3, 4$, to be couples of n.n. sites such that each one of them is not in Q_0^l but is n.n. of Q_0^l (see Fig. 6) and we set

$\tau_{l,i} = \inf\{k \geqslant 1$: I_k^2, I_{k+2}^2 are "good," I_{k+1}^2 is "bad" *only* because $\exists s \in I_{k+1}^2$ such that at time s we see the cut drawn in Fig. 6 (the dashed bonds) and $C_{\{x_i y_i\}} = +\}$

From the definition of good intervals I_k^2 we immediately have the following proposition:

Proposition 2.2. Let $|h| l^2 > 2$ and let σ be such that $\sigma(x) = -1$ $\forall x \in A \subset Q_0^l$ such that A enjoys property $\mathscr{P}$ of Lemma 2.1.

Then if $\tau_{l,1} = 1$, we have

$$\sigma_{3(4L_0)^2 T_2}(x) = -1 \qquad \forall x \in R_{l,l+1}$$

with $R_{l,l+1}$ as in Fig. 7.

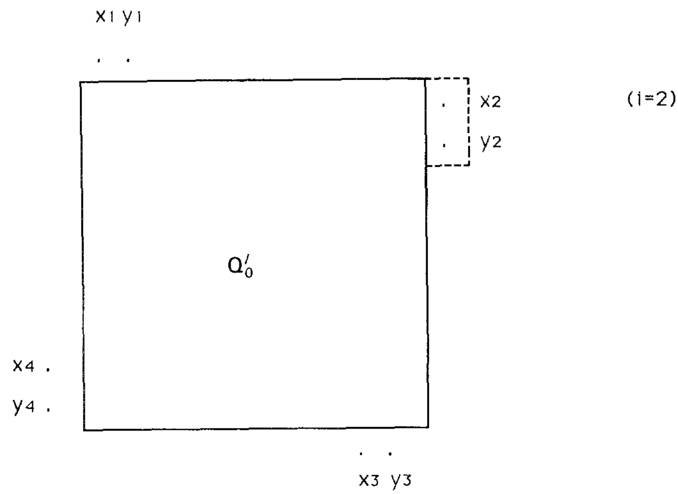


Fig. 6

From the explicit definition of $\tau_{l,i}$ it follows from a direct computation that the typical values of $\tau_{l,i}$ are of order $\exp(2\beta)$. This is in fact approximate, by the inverse of the probability of observing a cut of four bonds in a time interval of length T_2. This result, combined with (2.6), tells us that if the *resistance time* of a square Q^l becomes *larger* than $\tau_{l,i} \cdot T_2 \simeq \exp(4\beta)$, then the square Q^l of minuses has a greater tendency to grow than to shrink. One immediately finds that l should be such that

$$(|h| - 3\delta)(l-2) > 2$$

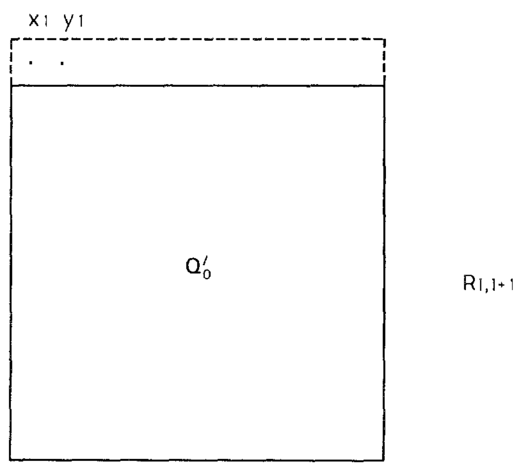


Fig. 7

i.e.,

$$l > l_c(h) \equiv \left[\frac{2}{|h|}\right] + 3 \tag{2.7}$$

if δ is sufficiently small.

We are finally in a position to construct the event $\mathscr{E}^{\text{sub}}(j)$.

We set:

(i) $\tau_0 = \inf\{k \geqslant 0$: at time $s_k = k(4L_0)^2 T_2 - 1$ we cut *all* the bonds between the square $Q_0^{[|h|^{-1/2}]\cdot 2}$ and its complement and any cluster C inside $Q_0^{[|h|^{-1/2}]\cdot 2}$ becomes minus one$\}$

$$\bar{\tau}_0 = T_2 \tau_0$$

(ii) $l_i = 2[2|h|^{-1/2}] + 2i,\ i = 0, 1,\ldots.$

(iii) $n_i = 3l_i,\ \gamma_i = l_i - 2,\ r_i = (l_i - 1)^2.$

(iv) $N_i = 4\exp\{\beta[2 \wedge (|h| - 3\delta)(l_i - 2) + \delta/2]\},\ N_{\text{tot}} = \sum_{i: l_i \leqslant l_c(h)} N_i.$

(v) $\sigma_i = \tau_0 + \sum_{0 \leqslant k \leqslant i} N_k T_2.$

(vi) $\mathscr{U}_i^0 = \{0 \leqslant \tau_{l_i,1} < N_i/4\} \cap \{N_i \leqslant \tau_{l_i,2} < N_i/2\} \cap \{N_i/2 \leqslant \tau_{l_i,3} < 3/4 N_i\} \cap \{3/4 N_i \leqslant \tau_{l_i,4} < N_i\} \cap \mathscr{R}_{n_i,\gamma_i,r_i}^{[0 N_i]\setminus\bigcup_{k=1}^4 [\tau_{l_i,k}, \tau_{l_i,k}+3]},\ i = 1, 2,\ldots$

(vii) $\mathscr{U}_i = S_{\sigma_{i-1}} \mathscr{U}_i^0$, where $i = 1, 2,\ldots,$
S_t is the shift operator defined in Section 1.

(viii) $\mathscr{E}_0^{\text{sub}} = \{\tau_0 = 0\} \cap \{I_{N_{\text{tot}}}^2 \text{ is good}\} \bigcap_{i: l_i \leqslant l_c(h)} \mathscr{U}_i.$

(ix) $\mathscr{E}^{\text{sub}}(j) = S_{(j-1) N_{\text{tot}} T_2 (4L_0)^2} \mathscr{E}_0^{\text{sub}},\ j > e^{2\beta}.$

The event $\mathscr{E}_0^{\text{sub}}$ can be understood in simple words. With $\tau_0 = 0$ we construct, for any configuration σ at time $t = 0$, a square $Q_0^{l_0}$ of minuses. Then the event $\mathscr{U}_1$ forces this square to grow to a square $Q_0^{l+2} \equiv Q_0^{l_1}$ within the time $N_1(4L_0)^2 T_2$ and to resist (up to small fluctuations) up to this time. In turn the event $\mathscr{U}_2$ enlarges $Q_0^{l_1}$ to $Q_0^{l_2}$ and so on.

Thus, it follows immediately from Lemma 2.1 and Propositions 2.1 and 2.2 that the event $\mathscr{E}^{\text{sub}}(j)$ satisfies the requirements (a), (b), (d) listed at the beginning of this section.

It remains to prove that

$$\lim_{\beta \to \infty} P(\exists e^{2\beta} \leqslant j \leqslant t_\beta/(4L_0)^2 T_2 \colon \mathscr{E}^{\text{sub}}(j) \text{ holds}) = 1 \tag{2.8}$$

This clearly follows if we can prove that

$$t_\beta e^{-2\beta - \delta\beta} P(\mathscr{E}_0^{\text{sub}}) \to \infty \tag{2.9}$$

when $\beta \to \infty$, because of the independence of the $\mathscr{E}^{\text{sub}}(j)$'s.

We have, by construction,

$$P(\mathscr{E}_0^{\mathrm{sub}}) = P(\tau_0 = 0)\, P(I^2_{N_{\mathrm{tot}}} \text{ is good}) \prod_{i: l_i \leqslant l_c(h)} P(\mathscr{U}_i) \tag{2.10}$$

Thus, we need an estimate from below of the probability of $S_{(j-1)N_{\mathrm{tot}}T_2(4L_0)^2}\mathscr{U}_i$, or, using translation invariance w.r.t. the time, of $\mathscr{U}_i^0$. We have

$$P(\mathscr{U}_i^0) \geqslant \sum_{\substack{N_i(j-1)/4 \leqslant k_j \leqslant N_i(i/4) \\ j=1,\dots,4}} P(\mathscr{U}_i^0 \cap \{\tau_{l_i,j} = k_j\}\, j = 1,\dots, 4) \tag{2.11}$$

Using the definitions of $\mathscr{U}_i^0$ and of $\tau_{l_i,j}$, we get that

$$\begin{aligned}
&\mathscr{U}_i^0 \cap \{\tau_{l_i,j} = k_j,\, j = 1,\dots, 4\} \\
&\quad = \bigcap \mathscr{R}_{n_i,\gamma_i,r_i}^{[0N_i]\setminus\bigcup_{j=1}^4 [k_j, k_j+3]} \\
&\qquad \bigcap \{\textstyle\bigcap_{j=1}^4 \{I^2_{k_j}, I^2_{k_l+2} \text{ are good; } I^2_{k_l+1} \text{ is bad only because } \exists s \in I^2_{k_j+1} \\
&\qquad \text{such that at time } s \text{ we have the cut } \overline{\dot{x}_j \quad \dot{y}_j} \text{ and } C_{\{x_j y_j\}} = +\}\}
\end{aligned} \tag{2.12}$$

Thus, using Lemma 2.2 and 2.3, we get, by explicit computation, that the rhs of (2.11) is bounded from below by

$$\begin{aligned}
&\sum_{\substack{[(j-1)/4]N_i \leqslant K_j \leqslant (j/4)N_i \\ j=1,\dots,4}} ((1-e^{-k\beta})^2 \{1 - [1 - e^{-4\beta}(1 - e^{-2\beta|h|})]^{(4L_0)^2 T_2}\})^4 \\
&\quad \times \{1 - [N_i e^{-k\beta n_i} + 2N_i n_i^2 e^{-2\beta - \beta|h|} \\
&\quad + N_i e^{-\beta(|h| - 2\delta)\gamma_i} + N_i (4L_0)^2\, T_2 k(L_0) e^{-|h|\beta\gamma_i}]\}
\end{aligned} \tag{2.13}$$

with $k = |h| - 2\delta$, which, in turn, after the insertion of the values r_i, n_i, N_i, γ_i, gives

$$e^{-8\beta + 4|h|l_i\beta - 8|h|\beta - 8\delta l_i\beta} \tag{2.14}$$

if $l_i \leqslant l_c(h)$.

Thus, using (2.10), we get

$$\begin{aligned}
P(\mathscr{E}_0^{\mathrm{sub}}) &\geqslant \left(\exp\frac{-8\beta}{|h|^{1/2}}\right) \prod_{i: l_i \leqslant l_c} \exp[-8\beta + 4|h|\, l_i\beta - 8\beta(|h| + \delta l_i)] \\
&\geqslant \exp\left[-\beta\left(\frac{4}{|h|} + c\right)\right]
\end{aligned} \tag{2.15}$$

where c is a constant independent of h if $h \leqslant 1$ and $\delta \approx h^2$.

Thus, if we take

$$t_\beta \equiv \exp\left(\frac{4\beta}{|h|} + \beta c\right) \tag{2.16}$$

we get (2.9).

The discussion of the subcritical regime is completed. ∎

3. CONSTRUCTION OF THE EVENT $\mathscr{E}_0$: THE SUPERCRITICAL REGIME

We complete here the construction of the event $\mathscr{E}_0$ by exhibiting an event $\mathscr{E}_j^{\text{super}}$ such that:

(i) The event $\mathscr{E}_0$ defined in Section 2 is $(0, 4L_0)$-cylindrical and $P(\mathscr{E}_0) \to 1$ as $\beta \to \infty$.

(ii) If $\{v_b(t), \xi_C(t)\} \in \mathscr{E}_j^{\text{super}} \cap \mathscr{B}_0$ for $j(4L_0)^2 T_2 \leqslant t \leqslant t_\beta$, $j \leqslant t_\beta e^{-\delta\beta - 2\beta}$, then, for any $\sigma \in \{-1, 1\}\Lambda$ such that

$$\sigma_{j(4L_0)^2 T_2}(x) = -1 \qquad \forall x \in Q_0^{l_c}$$

we have

$$\sigma_{t_\beta}(x) = -1 \qquad \forall x \in Q_0^{2L_0}$$

with $L_0 \gg l_c(h)$, e.g., $L_0 = 1/|h|^3$.

Clearly, it suffices to construct $\mathscr{E}_0^{\text{super}}$ and then set $\mathscr{E}_j^{\text{super}} \equiv S_{j(4L_0)^2 T_2} \mathscr{E}_0^{\text{super}}$.

The idea is very simple. If at time $t = 0$ we start with a square Q_0^{2l} of minuses supercritical, i.e., $l \geqslant l_c/2$, then the resistance time of this cluster (in the sense specified in Section 2) is at least

$$T_2 e^{2\beta + \delta\beta/2} \geqslant e^{4\beta + \delta\beta} \tag{3.1}$$

Thus, during this time it is extremely likely that we will be able to construct a protuberance like the one of Fig. 6 of Section 2 and thus enlarge the square Q_0^{l+2}. Iterating this argument $(L_0 - l)$ times, we reach a square $Q_0^{L_0}$ in a time of order

$$e^{4\beta + \delta\beta/2}$$

Once the droplet $Q_0^{L_0}$ is formed, if L_0 is chosen large enough, then it is an easy matter to show that it will be able to resist much longer than t_β.

We proceed now very much as in Section 2. We set

$$\begin{aligned} l_i &= 2\left[\frac{l_c}{2}\right] + 2i \\ N &= \exp(2\beta) \\ \gamma &= l_c(h) = \left[\frac{2}{|h|}\right] + 3 \\ n &= \frac{4}{|h| - 2\delta} \\ r &= \frac{3}{|h|} \end{aligned} \tag{3.2}$$

and we set

$$\mathscr{W}_i^0 = \{0 \leqslant \tau_{l_i j} < N\ \forall j = 1,..., 4\} \bigcap \mathscr{R}_{n,\gamma,r}^{[0N] \setminus \bigcup_{j=1}^4 [\tau_{l_i,j}, \tau_{l_ij} + 3]} \tag{3.3}$$

$$\mathscr{W}_i = S_{iN(4L_0)^2 T_2} \mathscr{W}_i^0 \tag{3.4}$$

As in Section 2, one checks immediately that $\mathscr{W}_i^0 \cap \mathscr{B}_0$ occurs and if σ is such that

$$\sigma(x) = -1 \qquad \forall x \in Q_0^{l_i} \tag{3.5}$$

then, if I_{N-1}^2 is "good,"

$$\sigma_{N(4L_0)^2 T_2}(x) = -1 \qquad \forall x \in Q_0^{l_i+1} \tag{3.6}$$

Thus, the event

$$\mathscr{W} = \bigcap_{i=0}^{L_0} \mathscr{W}_i \bigcap \{I_{L_0 N}^2 \text{ is good}\} \tag{3.7}$$

with $L_0 = 1/|h|^3$, $L_0 \ll L$, is such that $\mathscr{W} \cap \mathscr{B}_0$ implies that for any configuration σ with $\sigma(x) = -1\ \forall x \in Q_0^{l_c}$ at time $t = L_0 N (4L_0)^2 T_2$,

$$\sigma_t(x) = -1 \qquad \forall x \in Q_0^{2L_0} \tag{3.8}$$

It is very easy to check that, because of the choice (3.2) of the parameters l_i, N, γ, n, and r, one has

$$\lim_{\beta \to \infty} P(\mathscr{W}_i^0) = 1 \tag{3.9}$$

that is, using the independence of the events $\mathscr{W}_i$, $i=0, 1, ..., L_0$,

$$\lim_{\beta\to\infty} P(\mathscr{W})=1 \tag{3.10}$$

Notice that, as announced, the time scale involved here is

$$L_0(4L_0)^2 NT_2 \approx T_4$$

It remains to analyze the resistance of the droplet $Q_0^{2L_0}$ of minus spins.

What we have to prove is that this droplet is able to resist, up to small fluctuations, for a time much larger than the nucleation time t_β.

Since the nucleation time is much longer than T_4. We cannot use directly the event $\mathscr{R}_{n,\gamma,r}^{[0N]}$ with $N \simeq t_\beta/T_2$ because now its probability is no longer close to one. This is so because of the event $\mathscr{G}_n^{[0N]}$ (see Section 2); in fact, on a time scale t_β we will observe almost surely more than three contemporary cuts inside $Q_0^{4L_0}$.

In conclusion, we have to relax the conditions characterizing $\mathscr{R}_{n,\gamma,r}^{[0N]}$ in such a way that its probability becomes again close to one without, however, destroying the droplet $Q_0^{2L_0}$.

We set

$$\mathscr{N} = (\mathscr{F}_n^{[0N_0]})^c \cap (\mathscr{H}_{n,\gamma}^{[0,N_0]})^c \cap (\mathscr{M}_r^{[0N_0]})^c \tag{3.11}$$

with

$$\text{(i)} \qquad N_0 = t_\beta/(4L_0)^2 T_2 \tag{3.12}$$

$$\text{(ii)} \qquad r=n=\gamma=8/h^2$$

Using Lemma 2.3, one sees immediately that

$$\lim_{\beta\to+\infty} P(\mathscr{N})=1 \tag{3.13}$$

Proposition 3.1. Let $L_0 = 1/|h|^8$. Suppose that the event

$$\mathscr{N} \cap \mathscr{B}_0 \cap \{I_{N_0}^2 \text{ is good}\} \text{ holds}$$

Then, if $|h| \ll 1$, for any σ such that

$$\sigma(x) = -1 \qquad \forall x \in Q_0^{2L_0}$$

we have

$$\sigma_{t_\beta}(x) = -1 \qquad \forall x \in Q_0^{2L_0}$$

Assuming the proposition, let us complete the construction of the event

$$\mathscr{E}_0^{\text{super}} = \mathscr{W} \cap \{S_{L_0 N(4L_0)^2 T_2} \mathscr{N}\} \cap \{I^2_{L_0 N + N_0} \text{ is good}\} \tag{3.14}$$

Because of (3.10), (3.13), $\lim_{\beta \to +\infty} P(\mathscr{E}_0^{\text{super}}) = 1$ and because of (3.8) and Proposition 3.1, $\mathscr{E}_0^{\text{super}}$ also satisfies the second requirement stated at the beginning of the section.

Proof of Proposition 3.1. Let $S_n = I_j^2 \cup \cdots \cup I^2_{j+n}$, $0 \leqslant j+n \leqslant N_0$, $n \leqslant 8/|h|$ be a bad sequence and let us compute how many minus spins in $Q_0^{2L_0}$ can we lose during S_n starting from a droplet of minuses equal to $Q_0^{2L_0}$. We claim that the total loss during S_n cannot exceed $4\gamma r^2$. Let $s_1 \cdots s_k$, $k \leqslant \gamma \leqslant 8/h^2$, be the times in S_n at which we have at least two cuts and one cluster $C \subset Q_0^{4L_0}$, $C = +$, or more than two cuts. At each time s_i, $i = 1,\dots, k$, we can lose, by the definition of $\mathscr{N}$, at most r spins -1. For $t \in (s_i, s_{i+1})$ we lose only those clusters of minuses, of size less than or equal to r, that can be obtained with only one cut, or isolated clusters of size $\leqslant r$.

Let now $\bar{l} > \sqrt{r}$ and $2L_0/\bar{l}$ is an integer and let us partition the square $Q_0^{2L_0}$ into $(2L_0/\bar{l})^2$ squares $\bar{Q}_j$ of side $\bar{l}$. By construction, if at time $s_i + 1$ all the spins inside a given square $\bar{Q}_j$ happen to be minus one, then each one of them will not flip to $+1$ up to time s_{i+1}. Moreover, the number of squares $\bar{Q}_j$ such that

$$\sigma_{s_i}(x) = -1 \qquad \forall x \in \bar{Q}_j$$

and there exists a site $x_j \in \bar{Q}_j$ such that

$$\sigma_{s_i+1}(x_j) = +1$$

is obviously bounded by r.

Therefore, the total loss of minuses during the time interval s_n cannot exceed

$$\gamma r(\bar{l})^2 \leqslant 4\gamma r^2 < \frac{1}{|h|^7} \tag{3.15}$$

if $2L_0 > 2$ and $|h|$ is sufficiently small.

If we now choose $L_0 = 1/|h|^8$ with $|h|$ small enough, it is easy to check using (3.15) that at the end of the bad sequence S_n there exists in $Q_0^{2L_0}$ a subset A where the spins are minus one and such that property $\mathscr{P}$ of Lemma 2.1 applies to A. Thus, the good interval I^2_{j+n+1} is able to reconstruct the whole droplet $Q_0^{2L_0}$ of minuses. The proposition follows by applying the above argument to all the bad sequences in $[0, N_0]$. ∎

APPENDIX A

Proof of Lemma 1.1. Let

$$\mathscr{B}_x^c = \{\exists t \leqslant t_\beta;\ \text{at time } t\ \exists C = + \text{with } |C| \geqslant L_0 \text{ and } C \cap Q_{2L_0}^x \neq \varnothing\} \tag{A1}$$

Then we have

$$P(\mathscr{B}^c) \leqslant \sum_{k=n/4}^{n} \frac{1}{k!} \sum_{\substack{x_{i_1} \cdots x_{i_k} \in A \\ x_{i_l} \neq x_{i_k} \in A \\ \text{for } l \neq k}} P\left(\bigcap_{j=1}^{k} \mathscr{B}_{x_{i_j}}^c\right) \tag{A2}$$

If we can prove that

$$P\left(\bigcap_{j=1}^{k} \mathscr{B}_{x_{i_j}}^c\right) \leqslant \exp[-\lambda(\beta) k] \tag{A3}$$

$\forall k \geqslant n/4$ and $\lambda(\beta)$ a suitable constant, then (A2) may be bounded by

$$P(\mathscr{B}^c) \leqslant \sum_{k=n/4}^{n} \binom{n}{k} e^{-\lambda(\beta) k} \leqslant 2^n \exp[-\lambda(\beta) n/4] \tag{A4}$$

Thus we have to prove (A3) with

$$\lambda(\beta) = \frac{\beta |h| L_0}{4} - 2$$

Let

$$X_t = \#\{j \leqslant k;\ \exists C = + \text{at time } t,\ |C| \geqslant L_0,\ C \cap Q_{x_{i_j}}^{2L_0} \neq \varnothing\} \tag{A5}$$

We observe that if we denote by $\tilde{\mathscr{C}}_t$ the family of clusters C, $\tilde{\mathscr{C}}_t = \{C_l\}$, with the property that any $C_l = +$ at time t, $|C_l| \geqslant L_0$, $C_l \cap Q_{x_{i_j}}^{2L_0} \neq \varnothing$ for some $j = 1,..., k$ and $C_l \cap C_{l'} = \varnothing$ if $l \neq l'$, then the cardinality of $\tilde{\mathscr{C}}_t$ as a subset of A satisfies

$$|\tilde{\mathscr{C}}_t| = \sum_l |C_l| \geqslant L_0 X_t \tag{A6}$$

This is obvious since any $|C_l|$ is greater than L_0 and if a given C_l intersects exactly m squares $Q_{X_j}^{2L_0}$, $j = 1,..., m$, then if $L > 4L_0$,

$$|C_l| \geqslant mL - 1 \geqslant mL_0$$

Thus, with these considerations in mind, we estimate (A3) by

$$\sum_{\substack{0 \leqslant k_1 \leqslant k \\ \sum_{t \leqslant t_\beta} k_t \geqslant k}} \cdots \sum_{0 \leqslant k_{t_\beta} \leqslant k} \sum_{\substack{\mathscr{C}_1 \cdots \mathscr{C}_{t_\beta} \\ |\mathscr{C}_t| \geqslant L_0 k_t}} P(X_t = k_t; \tilde{\mathscr{C}}_t = \mathscr{C}_t\, \forall t \leqslant t_\beta) \tag{A7}$$

Because of the construction of the dynamics, (A7) in turn is bounded by

$$\sum_{\substack{k_1 \cdots k_{t_\beta} \\ \sum k_t \geqslant k}} \sum_{\substack{\mathscr{C}_1 \cdots \mathscr{C}_{t_\beta} \\ |\mathscr{C}_t| \geqslant L_0 k_t}} \exp\left(-\beta |h| \sum |\mathscr{C}_t|\right)$$

$$\leqslant \sum_{\substack{k_1 \cdots k_{t_\beta} \\ \sum k_t \geqslant k}} \exp\left(-\beta \frac{|h|}{2} L_0 \sum_{t=1}^{t_\beta} k_t\right) \left(\sum_{\mathscr{C}} \exp\left(-\beta \frac{|h|}{2} |\mathscr{C}|\right)\right)^{t_\beta} \tag{A8}$$

In order to evaluate the last sum, we expand again

$$\sum_{\mathscr{C}} \exp\left(-\beta \frac{|h|}{2} |\mathscr{C}|\right)$$

$$\leqslant \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\substack{x_{i_1} \cdots x_{i_l} \\ x_{i_l} \neq x_{i_{l'}} \\ l \neq l'}} \sum_{\substack{C_1 \cap Q^{2L_0}_{x_{i_1}} \neq \varnothing; |C_1| \geqslant L_0 \\ \vdots \\ C_l \cap Q^{2L_0}_{x_{i_l}} \neq \varnothing; |C_l| \geqslant L_0}} \exp\left(-\beta \frac{|h|}{2} \sum_j^l |C_j|\right) \tag{A9}$$

For $\beta |h|$ large enough, we have

$$\sum_{\substack{C \cap Q^{2L_0} \neq \varnothing \\ |C| \geqslant L_0}} \exp\left(-\beta \frac{|h|}{2} |C|\right) \leqslant (2L_0)^2 \exp\left(-\beta \frac{|h|}{2} L_0\right) \tag{A10}$$

which in turn gives for the rhs of (A9) the bound

$$1 + \exp\left(-\beta \frac{|h|}{4} L_0\right) (2L_0)^2 \tag{A11}$$

By inserting (A11) into (A8), we get that the rhs of (A8) is estimated by

$$\exp\left(-\beta \frac{|h|}{4} L_0 k\right) \left[\sum_{k=0}^{\infty} \exp\left(-\beta \frac{|h|}{4} L_0 k\right)\right]^{t_\beta} \left[1 + \exp\left(-\beta \frac{|h|}{4} L_0\right) (2L_0)^2\right]^{t_\beta} \tag{A12}$$

which is finally bounded by

$$\exp\left(-\beta \frac{|h|}{4} L_0 k + 2k\right) \tag{A13}$$

for β large enough and L_0 such that

$$t_\beta \exp\left(-\beta \frac{|h|}{4} L_0\right)(2L_0)^2 < 1$$

e.g., $L_0 \cong 1/|h|^3$, $0 < |h| \ll 1$.

The lemma is proved. ∎

Proof of Lemma 1.2. As in the proof of Lemma 1.1, we write

$$P(\mathscr{D}^c) \leqslant \sum_{m \geqslant n/4}^{n} \frac{1}{m!} \sum_{\substack{x_{i_1} \cdots x_{i_m} \\ x_{i_j} \neq x_{i_k} \\ k \neq j}} P\left(\bigcap_{j=1}^{m} \mathscr{E}^c_{x_{i_j}}\right)$$

If $p(\beta)$ denotes $1 - P(\mathscr{E}_x)$ $\forall x$, then we get

$$P(\mathscr{D}^c) \leqslant \sum_{m \geqslant n/4}^{n} \binom{n}{m} [p(\beta)]^m \leqslant \exp\left(-k(\beta)\frac{n}{4}\right)$$

for a suitable constant $k(\beta) \nearrow +\infty$ as $\beta \nearrow +\infty$, e.g., $k(\beta) = -\ln[2^4 p(\beta)]$. ∎

APPENDIX B

Proof of Lemma 2.3. Part (a) trivially follows from Lemma 2.2.

For part (b) we have

$$\begin{aligned} P(\mathscr{G}_n^{[0N]}) &\leqslant (N-n)\left[nT_2 \sum_{k \geqslant 4} e^{-\beta k} e^{-|h|\beta} k(L_0) + (nT_2)^2 e^{-6\beta - 2\beta|h|} k^2(L_0)\right] \\ &\leqslant 2Nn^2 k(L_0) e^{-2\beta - \beta|h|} \end{aligned}$$

if $\beta \gg 1$.

For part (c)

$$\begin{aligned} P(\mathscr{H}_{n,\gamma}^{[0N]}) &\leqslant \sup_j (N-n) \sum_{i=\gamma}^{n(4L_0)^2 T_2} \sum_{\substack{s_1 \cdots s_i \\ s_k \in I_j^2 \cup \cdots \cup I_{j+m}^2}} [2e^{-2\beta - |h|\beta} k(L_0)]^i \\ &\leqslant N \sum_{i=\gamma}^{\infty} [e^{-2\beta - \beta|h|} 2k(L_0)]^i [n(4L_0)^2 T_2]^i \\ &\leqslant N e^{-\beta(|h| - 2\delta)\gamma} \end{aligned}$$

if $\beta \gg 1$ and $0 < \delta < |h|/2$.

For part (d),

$$P(\mathscr{M}_r^{[0N]}) \leqslant NL_0^2 T_2 k(L_0) e^{-\beta|h|r}$$

trivially holds.

REFERENCES

1. R. H. Swendsen and J. S. Wang, Nonuniversal critical dynamics in Monte Carlo simulations, *Phys. Rev. Lett.* **58** (1987).
2. F. Martinelli, E. Olivieri, and E. Scoppola, On the Swendsen–Wang dynamics. I. Exponential convergence to equilibrium, *J. Stat. Phys.*, this issue; Rigorous analysis of low temperature stochastic Ising models: Metastability and exponential approach to equilibrium, Preprint, Rome (1989).
3. E. Jordao-Neves and R. Schonmann, Critical droplets and metastability for a Glauber dynamics at very low temperatures, Preprint, University of Sao Paulo (1989).